United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Thursday, 4 December 2008

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

2. Find all real values of x, y and z such that
$$(x+1)yz = 12,\ (y+1)zx = 4 \text{ and } (z+1)xy = 4.$$

3. Let $ABPC$ be a parallelogram such that ABC is an acute-angled triangle. The circumcircle of triangle ABC meets the line CP again at Q. Prove that $PQ = AC$ if, and only if, $\angle BAC = 60°$. *The circumcircle of a triangle is the circle which passes through its vertices.*

4. Find all positive integers n such that both $n + 2008$ divides $n^2 + 2008$ and $n + 2009$ divides $n^2 + 2009$.

5. Determine the sequences $a_0, a_1, a_2, \ldots$ which satisfy all of the following conditions:
a) $a_{n+1} = 2a_n^2 - 1$ for every integer $n \geq 0$,
b) a_0 is a rational number and
c) $a_i = a_j$ for some i, j with $i \neq j$.

6. The obtuse-angled triangle ABC has sides of length a, b and c opposite the angles $\angle A, \angle B$ and $\angle C$ respectively. Prove that
$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc.$$